Minimal Model Program Learning Seminar

Week 10:

- Multiplier Ideals.
- Examples.
- Nadel Vanishing Theorem.

Minimal Model Program:

Aim for today:

i) Introduce multiplier ideal sheaves, and

ii) Prove the Nadel Vanishing Theorem.

Theorem (Kawamata - Viehweg vanishing): X projective, (X,Δ) klt, M big & nef $\mathbb{Q}$-Cartier. Then

$$H^j(X, \mathcal{O}_X(K_X + \Delta + M)) = 0 \text{ for } j>0.$$

Question: Can we weaken any further:

- bigness. M. $\leftarrow$
- nefness. M $\leftarrow$
- kltness (X,Δ) $\leftarrow$

none of these can be weaken

Example: Projective cone over an elliptic curve. $H^i(\mathcal{O}_X) \neq 0$.

"Idea to weaken kltness":

$$\begin{tikzcd} Y \arrow[d, "\pi"] \\ (X,\Delta) \end{tikzcd}$$

log resolution of (X,Δ).

(X,Δ) log pair, M big & nef.

$$\pi^*(K_X+\Delta) = \underbrace{K_Y+\Delta_Y}_{\text{sub-klt}} + \underset{\substack{\vee\!\!\shortparallel \\ 0}}{E}$$

$$\pi^*(K_X+\Delta) = K_Y+\Gamma_Y \longrightarrow \text{could have coeff } \geq 1$$

We can't apply KV vanishing on $K_Y+\Gamma_Y$, but

$\underbrace{K_Y+\Delta_Y+\underbrace{\pi^*M}_{\text{big + nef}}}$ looks close enough to apply KV vanishing Thm

(*) Can we pull-back, apply KV vanishing on the log resolution, and then pushforward?

Multiplier ideals:

(X,Δ) log pair and V be a linear system on X.

Let $f: Y \longrightarrow X$ be a log resolution of (X,Δ) and V.

Write:

$$K_Y + \Gamma = f^*(K_X + \Delta) + E$$

$E, \Gamma \geq 0$ with disjoint support.

$0 \leq F = \mathrm{Fix}(f^*V)$ the fixed part of the linear system.

Fix $c > 0$, and we define the multiplier ideal of (X,Δ) with respect to $c \cdot V$:

$$\mathcal{J}((X,\Delta); c\cdot V) := f_* \mathcal{O}_Y(E - \lfloor cF \rfloor)$$

$$\mathcal{J}_{\Delta, cV} := f_* \mathcal{O}_Y(E - \lfloor cF \rfloor)$$

If no Δ, $\mathcal{J}(X; cV)$ or $\mathcal{J}_{cV}$.

If $D \geq 0$ $\mathbb{Q}$-Cartier and $m > 0$ such that mD is Cartier, then we let $\mathcal{J}_{\Delta, cD} := \mathcal{J}((X,\Delta); cD) := \mathcal{J}_{\Delta, \frac{c}{m}\cdot V}$..

where $V = \{mD\}$ is the linear system associated to mD.

Remark: If $\Delta = 0$ and $c = 1$, then the above definition is simply

$$\mu_* \mathcal{O}_Y(\underset{\substack{\shortparallel \\ K_Y - \mu^*(K_X)}}{K_{Y/X}} - [\mu^* D]).$$

Proposition: The definition of multiplier ideal is independent from the log resolution.

Proof:

$$\begin{tikzcd}
 & Y'' \arrow[dl] \arrow[dr] & \\
Y \arrow[dr] & & Y' \arrow[dl] \\
 & X &
\end{tikzcd}$$

Replace Y' by Y''.

We are in the following situation

$$\begin{tikzcd}
Y' \arrow[d, "\mu"] \arrow[dd, bend right, "f'"'] \\
Y \arrow[d, "f"] \\
X
\end{tikzcd}$$

We are in the following situation
$$\begin{tikzcd} Y' \arrow[d, "\mu"] \arrow[dd, bend right, "f'"'] \\ Y \arrow[d, "f"] \\ X \end{tikzcd}$$

Both f and f' are log resolutions of (X, Δ) and V.

$$K_Y + \Gamma = f^*(K_X + \Delta) + E, \qquad K_{Y'} + \Gamma' = (f')^*(K_X + \Delta) + E'.$$

$$K_{Y'} + \Gamma' = \mu^*(K_Y + \Gamma - E) + E'.$$

$F = \mathrm{Fix}(f^* V)$ and $F' = \mathrm{Fix}((f')^* V)$.

By definition $f^* V - F$ is bpf, then $F' = \mu^* F$.

Claim: $\mu_* \mathcal{O}_{Y'}(E' - \mu^* E - \lfloor \mu^* \{cF\} \rfloor) = \mathcal{O}_Y$.

Proof: $K_Y + \Gamma + \{cF\}$ is dlt and has the same log canonical places than $K_Y + \Gamma - E$; furthermore, we have that

$$K_{Y'} + \Gamma' + \mu^*(E + \{cF\}) - E' = \mu^*(K_Y + \Gamma + \{cF\})$$

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$$K_{Y'} + \Gamma' + \underbrace{\mu^*(E + \{cF\}) - E'} = \mu^*(K_Y + \Gamma + \{cF\})$$

It follows that $\lfloor \mu^*(E + \{cF\}) - E' \rfloor \leq 0$, hence $E' - \mu^* E - \lfloor \mu^* \{cF\} \rfloor$ is eff and μ-exc.

Thus, $\mu_* \mathcal{O}_{Y'}(E' - \mu^* E - \lfloor \mu^* \{cF\} \rfloor) = \mathcal{O}_Y.$ $\square$

Return to the proof:

$f'_* = f_* \mu_*$

$f' = \mu \circ f$

$$(f')_* \mathcal{O}_{Y'}(E' - \lfloor cF' \rfloor) =$$

proj formula for μ.

$$(f')_* \mathcal{O}_{Y'}(E' - \mu^* E - \lfloor \mu^* \{cF\} \rfloor + \mu^*(E - \lfloor cF \rfloor)) =$$

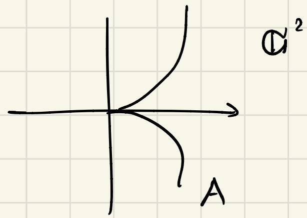
$$f_*(\underbrace{\mu_* \mathcal{O}_{Y'}(E' - \mu^* E - \lfloor \mu^* \{cF\} \rfloor)}_{= \mathcal{O}_Y} \otimes \mathcal{O}_Y(E - \lfloor cF \rfloor)) =$$

$$= f_*(\mathcal{O}_Y(E - \lfloor cF \rfloor))$$

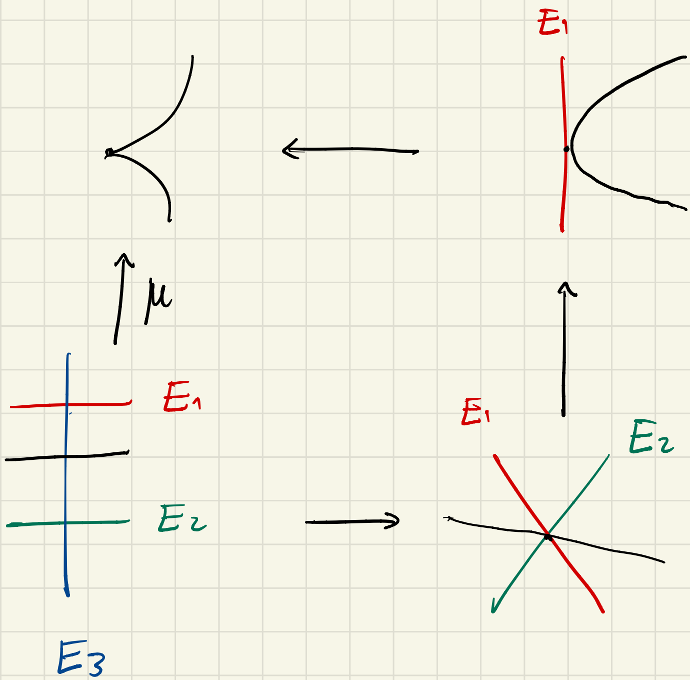
Example: If D is Cartier on X, then

$$\mathcal{J}_D = \mathcal{O}_X(-D).$$

Example: $X = \mathbb{C}^2$, $A \subseteq \mathbb{C}^2$ cuspidal curve $A = \{y^2 - x^3\}$



$\mathcal{J}(\frac{5}{6}A) = \langle x, y \rangle$ $\qquad$ $\mathcal{J}(cA) = \mathcal{O}_X$ $\quad 0 < c < \frac{5}{6}$.



$$\mu^*(A) = A' + 2E_1 + 3E_2 + 6E_3.$$

Example: $X = \mathbb{C}^n$, $A \subseteq X$ a smooth divisor with an ordinary $(n+1)$-fold point at the origin.

$$\mathcal{J}(\mathbb{C}^n, \tfrac{n}{n+1} A) = m_0.$$

$\pi : X \longrightarrow \mathbb{C}^n$ blow-up of $\mathbb{C}^n$ at the origin and compute

$\pi^*(A) = A_{X'} + (n+1)E$ and $\pi^*(K_{\mathbb{C}^n}) = K_X - nE.$

Proposition (Bertini-Kollár): $D \geq 0$ $\mathbb{Q}$-divisor on a smooth variety X. $|V|$ free linear system. $A \in |V|$ general. Then $\mathcal{J}(D + cA) = \mathcal{J}(D)$. for $0 < c < 1$.

Proposition (Adding integral divisors): Let X be a smooth variety, A be an integral divisor, and D be an eff $\mathbb{Q}$-div. Then $\mathcal{J}(D + A) = \mathcal{J}(D) \otimes \mathcal{O}_X(-A)$

Proposition (Adding integral divisors): Let X be a smooth variety, A be an integral divisor, and D be an eff $\mathbb{Q}$-div. Then $\mathcal{J}(D+A) = \mathcal{J}(D) \otimes \mathcal{O}_X(-A)$

Proof: $X' \xrightarrow{\mu} X$ log resolution of (X, D).

$$[\mu^*(D+A)] = [\mu^* D + \mu^* A] = [\mu^* D] + \mu^* A.$$

Therefore, we have

$$\mu_* \mathcal{O}_{X'}(K_{X'/X} - [\mu^*(D+A)]) =$$

$$\mu_*(\mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) \otimes \mathcal{O}_{X'}(-\mu^* A)) =$$ proj formula

$$\mu_* \mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) \otimes \mathcal{O}_X(-A)$$

$$\mathcal{J}(D) \otimes \mathcal{O}_X(-A)$$

$\square$

Multiplier ideals and singularities:

Proposition: X of dimension n, $D \geq 0$ $\mathbb{Q}$-divisor.
$\mathrm{mult}_x D \geq n$ for some smooth point $x \in X$ (closed). Then $\mathcal{J}(D) \subseteq m_x$
More generally, if $\mathrm{mult}_x D \geq n+p-1$. Then $\mathcal{J}(D) \subseteq m_x^p$.

Proof: $X' \xrightarrow{\mu} X$ blow-up of x.

$\mathrm{ord}_E(K_{X'/X}) = n-1$, $\mathrm{ord}_E(\mu^* D) = \mathrm{mult}_x D$.

On the other hand $\mu_* \mathcal{O}_{X'}(-nE) = m_x^n$.

If $\mathrm{mult}_x D \geqslant n+p-1$, then

$$\mathrm{ord}_E(K_{X'/X} - [\mu^* D]) \leqslant -p.$$

Hence $\mathcal{J}(D) \subseteq \mu_* \mathcal{O}_{X'}(-pE) = m_x^p$.

we get the containment $\mathcal{J}(D) \subseteq m_x^p$.

Proposition: (X,D) log canonical iff $\mathcal{J}(X,(1-\varepsilon)D) = \mathcal{O}_X$
$0 < \varepsilon < 1$

(X,D) klt iff $\mathcal{J}(X,D) = \mathcal{O}_X$.

Monomial ideals and Newton polytopes:

$$\mathfrak{a} = \langle x^{m_1}, \dots, x^{m_\ell} \rangle \subseteq \mathbb{K}[x_1, \dots, x_n]$$

$$x^{m_i} = x_1^{m_{i,1}} x_2^{m_{i,2}} \cdots x_n^{m_{i,n}}$$

$$P(\mathfrak{a}) := \text{Convex hull}(\underbrace{m_1, \dots, m_\ell}_{\subseteq \mathbb{Q}^n}) + \mathbb{Q}^n_{\geq 0}$$

Minkowski sum

$P(\mathfrak{a})$ is called the Newton polytope associated to $\mathfrak{a}$.

$f = \sum' c_m x^m \qquad \mathfrak{a} = \langle x^m \mid c_m \neq 0 \rangle.$

what $P(\mathfrak{a})$ says about f?

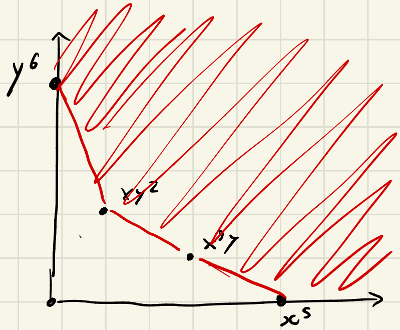
$f + t x^{m_0}$ How the family of sing changes?

"Essentially" the sing does not change iff $m_0 \in \operatorname{int}(P(\mathfrak{a}))$

$\operatorname{div}(f + t x^{m_0}) = D_t$, $\mathcal{J}(\mathbb{C}^n, c D_t)$ does not depend on t as far as $x^{m_0} \in \operatorname{int}(P(\mathfrak{a}))$.

Remark: Mild condition on f called non-degeneracy of Newton pol.

Example: $\mathfrak{a} = \langle y^6, xy^2, x^3y, x^7 \rangle$



Theorem (Howald's): The multiplier ideal $\mathcal{J}(c\cdot\mathfrak{a})$ is the monomial ideal generated by x^v so that

$$v + \underset{\substack{\parallel \\ (1,\dots,1)}}{\mathbf{1}} \in \operatorname{int}(c \cdot P(\mathfrak{a})).$$

Lemma: $f : V \to W$ proper morphism, A ample on W. $\mathcal{F}$ coherent on V. Assume $H^j(V, \mathcal{F} \otimes \mathcal{O}_V(f^* mA)) = 0$ for $j > 0$ and $m \gg 0$. Then $R^j f_* \mathcal{F} = 0$ for $j > 0$.

Proof: Pick $m \gg 0$ so that

1.- $H^i(W, R^j f_* \mathcal{F} \otimes \mathcal{O}_W(mA)) = 0$ for $i > 0$, $j \geq 0$. by Serre vanishing

2. $R^j f_* \mathcal{F} \otimes \mathcal{O}_W(mA)$ is non-zero and g.g. as long as $R^j f_* \mathcal{F}$ is non-trivial. Serre vanishing.

By the Leray spectral sequence.

Leray spectral

$$\underbrace{H^j(V, \mathcal{F} \otimes \mathcal{O}_V(f^* mA))}_{\substack{\| \\ 0 \\ \text{by assumption}}} = H^0(W, \underbrace{R^j f_* \mathcal{F} \otimes \mathcal{O}_W(mA)}_{\text{g.g. as long as } R^j f_* \mathcal{F} \neq 0}).$$

Then, we conclude that $R^j f_* \mathcal{F} = 0$

Theorem (Local vanishing): X smooth, $D \geq 0$ $\mathbb{Q}$-divisor. Let $X' \xrightarrow{\mu} X$ be a log resolution of (X, D). Then

$$R^j \mu_* \mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) = 0 \quad \text{for } j \geq 0.$$

Proof: Assume X is projective, A ample so that $A - D$ amp.

$\mu^*(A-D)$ nef & big.

KV vanishing $\quad H^j(X', \mathcal{O}_{X'}(K_{X'} + \mu^* A - [\mu^* D])) = 0$ for $j > 0$

Hence, by the Lemma:

$\mu^* K_X - \mu^* K_X$ + proj form.

$$R^j \mu_* \mathcal{O}_{X'}(K_{X'} + \mu^* A - [\mu^* D]) =$$

$$R^j \mu_* \mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) \otimes \mathcal{O}_X(K_X + A) = 0$$

for every $j > 0$, so we conclude that the left side is zero.

Theorem (Nadel vanishing): Let X be a smooth proj

$D \geq 0$ $\mathbb{Q}$-divisor. L Cartier divisor such that $L-D$

is nef and big. Then,

$$H^i(X, \mathcal{O}_X(K_X+L) \otimes \mathcal{J}(D)) = 0 \quad , \text{for } i > 0$$

Rmk: If (X, D) is klt, we recover KV vanishing.

Proof: $X' \longrightarrow X$ log resolution of (X, D)

By KV vanishing

$$H^i(X', \mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) \otimes \mu^* \mathcal{O}_X(K_X+L))$$

$$= H^i(X', \mathcal{O}_{X'}(K_{X'} + \mu^* L - [\mu^* D])) = 0$$

for $i > 0$

big & nef.

By local vanishing, we have that

$$R^j \mu_* (\mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) \otimes \mu^* \mathcal{O}_X(K_X+L)) =$$

$$R^j \mu_* (\mathcal{O}_{X'}(K_{X'/X} - [\mu^* D])) \otimes \mathcal{O}_X(K_X+L) = 0 \quad \text{for } j > 0.$$

Finally, we have that:

$$\mu_*(\mathcal{O}_{X'}(K_{X'/X}-[\mu^*D])\otimes\mu^*(\mathcal{O}_X(K_X+L))=$$

$$\mu_*(\mathcal{O}_{X'}(K_{X'/X}-[\mu^*D]))\otimes\mathcal{O}_X(K_X+L)=$$

$$\mathcal{O}_X(K_X+L)\otimes\mathcal{J}(D).$$

Then, by Leray spectral sequence

$$H^i(X,\mathcal{O}_X(K_X+L)\otimes\mathcal{J}(D))=$$

vanish $R^j\mu_*=0$

$$H^i(X,\mathcal{O}_{X'}(K_{X'/X}-[\mu^*D]+\mu^*(K_X+L)))=0 \quad \text{for } i>0$$

KV.

Remark:

$$\omega_D\otimes\mathcal{O}_D(-D) \text{ is } \omega_X|_D$$

$$0\longrightarrow\omega_X(-D)\longrightarrow\omega_X\longrightarrow\omega_X|_D\longrightarrow 0$$

Lifting sections with KV:

$i>0$ $H^i(\omega_X(-D))=0$ to prove

$$H^0(\omega_X)\twoheadrightarrow H^0(\omega_D\otimes\mathcal{O}_D(-D)).$$

Lifting section with Nadel:

$$0\longrightarrow\omega_X\otimes\mathcal{J}(D)\longrightarrow\omega_X\longrightarrow\omega_X|_{V(\mathcal{J}(D))}\longrightarrow 0.$$

$$H^i(\omega_X\otimes\mathcal{J}(D))=0$$

$$H^0(\omega_X)\longrightarrow H^0(\omega_X|_{V(\mathcal{J}(D))})$$